Weak Extension Principle

Deniz Yılmaz

supervised by

Alessandro Vignati

Master's thesis
Logique Mathématique et Fondements de l'Informatique
September 2023

Contents

Acknowledgements

I would like to express my deepest gratitude to Alessandro Vignati for his invaluable guidance and support throughout my master's studies. He supervised two works of mine: this report as well as a first year report entitled "Ultrafilters in Ramsey Theory".

I am grateful to the Institut National des Sciences Mathématiques et de leurs Interactions of CNRS for providing me with the opportunity to receive a funding for this work.

I would also like to thank my family and friends for their support and patience during this process.

Special thanks to Adrien Deloro for his guidance throughout my master's studies.

Finally many thanks to all my teachers, especially to Mirna Džamonja and Boban Veličković. Their courses and this year at LMFI brought me a lot.

Deniz Yılmaz,
Paris, August 2023.

Chapter 1

Introduction

This report is in set theory, more specifically at the intersection between set-theoretic topology and Boolean algebras. Its main goal is to study homeomorphisms of Stone-Čech remainders of zero-dimensional locally compact Polish spaces. The origin of the theme lies in the study of Boolean algebra automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$, where ω is the discrete countable space and $\mathrm{Fin} \subseteq \mathcal{P}(\omega)$ is the ideal of finite subsets. The subject has further been developed for non-commutative operator algebras by Ilijas Farah and his collaborators. The introduction has three sections which contain respectively the contents of our work, the setting of the problem, and its historical development.

1.1 Contents of the report

The goal of this work is understanding continuous maps, say φ, between Stone-Čech remainders, and how these behave in terms of continuous maps between the underlying spaces. We will call φ *trivial*, if it can be defined in terms of a map between the underlying spaces in a natural way. Our principal question is the following: Are all homeomorphisms between Stone-Čech remainders trivial?

The main content of this report summarises some of the results of [FM12] and [Vel93]. Therefore we had to first assimilate a number of extra-curricular topics such as the Open Coloring Axiom, Parovičenko's theorem on corona spaces, and some infinite combinatorics. Sources are indicated below. As a result, the first half of the internship was dedicated to these prerequisites, and only the second half to reading research papers. However the report is in reverse order. It focuses on [FM12] and [Vel93], with all the interesting groundwork collected in final appendices.

- Appendix A is a survey on Martin's Axiom ($\mathsf{MA}_{\aleph_1}$; see Appendix A.1), Open Coloring Axiom (OCA; see Appendix A.2) and infinitary combinatorics. For preparing this chapter we studied related chapters of [Tod89] and [Far19].

- Appendix B is a review of the Stone-Čech compactification, Stone duality and their relation for zero-dimensional, locally compact spaces. We studied related chapters of [CN74] and [Wal74] and quoted the required results there. In this appendix we also studied Parovičenko's characterization of ω^* which leads to a negative answer for our problem as we mentioned earlier. For this we referred to [KV84, Chapter 11].

Then we move to [FM12]. Farah and McKenney show the following:

> Assume $OCA + \mathsf{MA}_{\aleph_1}$. Let X, Y be zero-dimensional, locally compact, non compact, Polish spaces. Then every homeomorphism $\varphi : X^* \to Y^*$ is trivial.

We can divide the proof of this theorem into two: showing that embeddings of $\mathcal{P}(\omega)/\mathrm{Fin} \to \mathcal{C}(X)/\mathcal{K}(X)$ are trivial under $\mathsf{MA}_{\aleph_1} + \mathsf{OCA}$ and amalgamating these trivial embeddings to construct trivial homeomorphisms. The present report covers the first part of this proof.

We first study the results for definable (Borel) automorphisms; this is done in ZFC. Then we study the P-ideals and tree-like families for a proof of "embeddings are trivial". All proofs in this report follow the "blueprint" introduced by Veličković in [Vel93]. In fact this "blueprint" is the one that gets used over and over even when treating more complicated objects and their automorphisms.

1.2 Setting of the problem

We shall start this section with some words on the motivation of this work. Topology is a branch of mathematics which has deep roots. Topological ideas are used nowadays roughly in every domain of mathematics since it aims to generalise what we know occurs in the reals on more general metric spaces. It has been in great interaction with set theory historically that the literature on set-theoretic topology is considerably large.

An embedding of a topological space X as a dense subset of a compact space is called a *compactification* of X. It is often useful to embed topological spaces in compact spaces, because of the special properties that compact spaces have. Stone-Čech is the largest compact space in which a locally compact space X sits densely, which factorises all continuous maps from X into a compact space K as a universal property.

One can not see much in compactification itself, yet the remainder is much more flexible. Therefore corona spaces becomes an exciting place where topologists, set theorists, infinite combinatorists, Boolean algebraists and analysts meet.

We study classical subclasses of the class of completely regular spaces; X will stand for one such space. Denote the Stone-Čech remainder (also called *corona space*) $\beta X \setminus X$ by X^*. Our main objects of interest will be continuous maps $\varphi : X^* \to Y^*$. We would like to understand them in terms of continuous functions F between the underlying spaces X and Y which extend to X^* and Y^*. Ideally for all φ we would like to find such F. This would decrease the complexity of the study of continuous functions between corona spaces. This amounts to understanding what information about X^* is encoded by X itself. For that purpose we will use the following notion: φ is called *trivial*, if there exists a continuous $F : X \to Y$ such that $\varphi = \beta F \setminus F$ where βF is the unique continuous extension of F to βX. Note that existence of these functions does not depend on set theory, as involving only countable information between Polish spaces. In other words they are absolute. Details on the Stone-Čech compactification can be found in Appendix B.1.

We will focus on the zero-dimensional case. More precisely, the space X under study will always be zero-dimensional, locally compact, non compact and Polish. In this context, Stone duality (see Appendix B.2) provides a correspondence between the topological notions in the left column and their Boolean algebraic counterparts in the right column:

βX	$\mathcal{C}(X)$
X^*	$\mathcal{C}(X)/\mathcal{K}(X)$
$\varphi : X^* \to Y^*$	$\varphi^d : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$

where $\mathcal{C}(X)$ denotes the Boolean algebra of clopen sets of X, $\mathcal{K}(X) \subseteq \mathcal{C}(X)$ is the compact-open ideal and φ^d is the corresponding Boolean algebra homomorphism. Therefore one can work with Boolean algebras and their homomorphisms instead of working directly in the topological setting which is arguably more difficult to handle. The prototype of structures in the scope of this work is the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ which is the dual of ω^* where ω is the discrete space of integers given by the usual order topology. Pioneering work in this domain has been made on the automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$. We will mention it in more detail below.

The following question will be the main interest of this report:

> Are all homeomorphisms $\varphi : X^* \to Y^*$ trivial?

We will see that an answer to this question depends on the set theoretical ambient.

1.3 Development of the topic

The Continuum Hypothesis (CH) implies that Stone-Čech remainders of zero-dimensional, locally compact, non compact and Polish spaces are all homeomorphic. In particular they are homeomorphic to ω^*. This is a consequence of Parovičenko's characterisation of ω^* from his 1963 paper [Par63] (for some details see Appendix B.3). From Rudin's 1956 paper [Rud56] we know that under CH we can construct a nontrivial automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$. Therefore CH produces a negative answer to our main question above.

The first positive answer to the question restricted to ω^* is given by Shelah in [She82]. There he shows that "all automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ are trivial" is consistent with ZFC. This arguably "difficult" proof uses the oracle chain condition. Then Shelah and Steprāns in their 1988 paper [SS88]. In this paper they show that the Proper Forcing Axiom (PFA) implies that every automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$ is trivial. One can refer to survey paper [FGVV22] for details of Shelah's construction.

$\mathsf{MA}_{\aleph_1}$ was introduced by Donald A. Martin and Robert M. Solovay in their 1970 paper [MS70]. This axiom has been very fruitful by giving numerous interesting combinatorial, analytic and topological consequences (see [Fre08]). Todorčević synthesised the OCA and proved its relative consistency from PFA in his 1989 monography [Tod89]. OCA is an uncountable generalisation of the Baire category theorem, and is known to contradict CH as $\mathsf{MA}_{\aleph_1}$. These Ramsey type axioms are used to study the automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ by Veličković in his 1990 article [Vel93]. There Veličković proved that every automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$ is trivial from another perspective.

Farah extened the study of the "behaviour under $\mathsf{OCA}+\mathsf{MA}_{\aleph_1}$" to different spaces, and gave a new prospective to Veličković's theorem by stating and proving the OCA lifting theorem in his monograph [Far00]. He also systematised the problem by introducing the *weak extension principle* $wEP(X,Y)$.

$wEP(X,Y)$ is the following statement:

> For any continuous function $F : X^* \to Y^*$, there exists a clopen partition $X^* = U_0 \cup U_1$ such that $F[U_0]$ is nowhere dense in Y^* and $F \restriction U_1$ lifts to a continuous $F_1 : \beta X \to \beta Y$ where $F_1[X] \subseteq Y$.

Farah and McKenney showed in their 2018 paper [FM12] the following: $\mathsf{OCA}+\mathsf{MA}_{\aleph_1}$ implies that homeomorphisms between Stone-Čech remainders of zero-dimensional, locally compact, non compact, Polish spaces are trivial.

This analysis was brought to the noncommutative setting again by the pioneering work of Farah, who showed that OCA implies all automorphisms of the Calkin algebra are inner, and by further work of his students. A survey of the current state of the art is [FGVV22].

Chapter 2

General Notation

In this chapter we will introduce some basic definitions which will be present throughout the text and derive some elementary results. The reader who is familiar with the subject may safely skip this chapter. Remind that X is always assumed as a zero-dimensional, locally compact, non compact, Polish (in other words separable and completely metrisable) space unless stated otherwise.

Notation 2.0.1. $\mathcal{C}(X)$ will denote the *algebra of clopen subsets* of X and $\mathcal{K}(X)$ will denote its *ideal of compact-open sets.*

Proposition 2.0.2. *There is an increasing sequence of compact-open sets* $\langle K_n : n \in \omega \rangle$ *such that* $X = \bigcup_{n\in\omega} K_n$.

Proof. X is Polish and therefore it is second countable. Then there is a countable clopen basis $\mathcal{B} = \{B_n : n \in \omega\}$. By locally compactness every point $x \in X$ has an open neighbourhood V_x such that $cl(V_x)$ is compact. Then for every $x \in X$, there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq V_x$. Therefore we have $\mathsf{cl}(B_x) = B_x$ is compact for all x and $X = \bigcup_{x\in X} \mathsf{cl}(B_x)$. Yet $\mathcal{B}$ is countable so there is a countable $I \subseteq X$ such that $X = \bigcup_{x\in I} B_x$. By identifying I with ω and defining $K_n = \bigcup_{i\leq n} \mathsf{cl}(B_i)$ we finish the proof. $\square$

Observation 2.0.3. *Such an increasing sequence* $\langle K_n : n \in \omega \rangle$ *forms a basis of* $\mathcal{K}(X)$ *as for all* $K \in \mathcal{K}(X)$*, there is some* $n \in \omega$ *such that* $K \subseteq K_n$.

Proof. Let $K \subseteq X$ a compact subspace. Then $\{K_n \cap K : n \in \omega\}$ is an open cover of K. By compactness of K, there is some $n \in \omega$ such that $K = K_n \cap K$ since K_n is an increasing sequence. $\square$

Let X and Y be two spaces carrying an algebraic structure compatible with the topology. Let $I \subseteq X$ and $J \subseteq Y$ be two definable (e.g., Borel, or analytic) substructures inducing quotients X/I and Y/J. Let $\varphi : X/I \to Y/J$ be a morphism. A *lifting* is a morphism $\varphi^* : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi^*} & Y \\ \big\downarrow{\scriptstyle \pi_I} & & \big\downarrow{\scriptstyle \pi_J} \\ X/I & \xrightarrow{\varphi} & Y/J \end{array}$$

commutes.

When X and Y are zero-dimensional, locally compact Polish spaces, by Stone Duality B.2, the dual of X^* is $\mathcal{C}(X)/\mathcal{K}(X)$ and the dual of a continuous map $\varphi : X^* \to Y^*$ is a Boolean algebra homomorphism $\varphi^d : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$.

Definition 2.0.4. Let $e : X \to \omega$ be a continuous map. Then e is called *compact-to-one* if $e^{-1}(n) \in \mathcal{K}(X)$ for all $n \in \omega$.

Observation 2.0.5. *Let $e : X \to \omega$ be compact-to-one. Then the homomorphism defined by*

$$\begin{aligned} F_e : \mathcal{P}(\omega) &\to \mathcal{C}(X) \\ a &\mapsto e^{-1}(a) \end{aligned}$$

induces canonically a homomorphism $\varphi_e : \mathcal{P}(\omega)/fin \to \mathcal{C}(X)/\mathcal{K}(X)$ such that

$$\begin{array}{ccc} \mathcal{P}(\omega) & \xrightarrow{F_e} & \mathcal{C}(X) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{P}(\omega)/\mathrm{Fin} & \xrightarrow{\varphi_e} & \mathcal{C}(X)/\mathcal{K}(X) \end{array}$$

commutes.

Proposition 2.0.6. *φ_e is injective if and only if e is bounded on compacts.*

Proof. Suppose that e is bounded on compacts. Let $a, b \in \mathcal{P}(\omega)$ such that $\varphi_e(a) = \varphi_e(b)$ i.e $e^{-1}(a) \triangle e^{-1}(b) \in \mathcal{K}(X)$. Then $e^{-1}(a \triangle b) \in \mathcal{K}(X)$. Therefore by the assumption $a \triangle b \in \mathrm{Fin}$.

For the converse, suppose for a contradiction that φ_e is injective and there is $K \in \mathcal{K}(X)$ such that e is not bounded on K. Define the increasing sequence of compacts $\langle K_n : n \in \omega \rangle$ by $K_n = \bigcup\{e^{-1}(i) : i \leq n\}$. Then $X = \bigcup_{n \in \omega} K_n$. So there is some $n \in \omega$ such that $K \subseteq K_n$. Let $a = e(K)$. By the hypothesis a is infinite and for all $i \in a \backslash n$ we have $e^{-1}(i) = \emptyset$ since $K \subseteq K_n$. We have therefore $\varphi_e(a \setminus n) = \emptyset$. Since φ_e is injective, $a \setminus n \in \mathrm{Fin}$ which is contradictory. □

Definition 2.0.7. A homomorphism $\varphi : \mathcal{P}(\omega)/\mathrm{Fin} \to \mathcal{C}(X)/\mathcal{K}(X)$ is called *trivial* if there exists a continuous compact-to-one $e : X \to \omega$ such that $\varphi = \varphi_e^*$.

Observation 2.0.8. *In the case $\varphi \in \mathrm{Aut}(\mathcal{P}(\omega)/\mathrm{Fin})$ triviality is equivalent to existence of a bijection $e : \omega \setminus a \to \omega \setminus b$ for some $a, b \in \mathrm{Fin}$ such that $\varphi = \varphi_e$.*

Chapter 3

Borel Liftings

In this chapter we will give proofs of some classical Borel lifting theorems. All the results are proven in the ZFC without need of any additional axioms. Throught this chapter we identify $\mathcal{P}(\omega)$ with 2^ω by mapping $x \subseteq \omega$ to its characteristic function χ_x. This endows $\mathcal{P}(\omega)$ with the product topology and the resulting space is called the Cantor space.

3.1 Borel automorphisms

Theorem 3.1.1 (Theorem 1.1 in [Vel93])**.** *Let $\varphi \in \mathrm{Aut}(\mathcal{P}(\omega)/\mathrm{Fin})$. Suppose that there is a dense G_δ subset $X \subseteq \mathcal{P}(\omega)$ and a continuous function $F : X \to \mathcal{P}(\omega)$ such that $\varphi[a] = [F(a)]$ for all $a \in X$. Then φ is trivial.*

Proof. We will first consider that $X = 2^\omega$, then generalise the result i.e φ has a continuous lifting. Denote the standard opens of 2^ω as N_s where $s \in 2^{<\omega}$. For $s, t \in 2^{<\omega}$ we say that s *forces* t if $F(N_s) \subseteq N_t$. Construct by induction the increasing integer sequence $\langle n_i : i \in \omega \rangle$ and functions $f_i : [n_i, n_{i+1}) \to 2$ such that

1. for all $i \in \omega$ and all $s \in 2^{n_i}$ we have $s \cup f_i$ forces some $t \in 2^{n_i}$;
2. for all $i \in \omega$, $s, s' \in 2^{n_i}$, $k > n_{i+1}$ and $g : [n_{i+1}, k) \to 2$ if $s \cup f_i \cup g$ forces some t and $s' \cup f_i \cup g$ forces some t' then $t(j) = t'(j)$ for all $j \geq n_{i+1}$.

Claim 3.1.2. Above sequence and functions are well defined.

Proof. Suppose that we have f_{i-1} and n_i. We will to construct f_i and n_{i+1} following (1) and (2).

Let $\{s_j : j \in 2^{n_i}\}$ be an enumeration of 2^{n_i}. Define partial functions f_i^j for $j \in 2^{n_i}$ inductively as follows: $f_i^0 = f_{i-1}$. Suppose f_i^j is defined and $\mathsf{dom} f_i^j \cap n_i = \emptyset$. Since F is continuous and $N_{s_j \cup f_i^j}$ is compact we would find eventually an $f_i^{j+1} \supseteq f_i^j$ such that $s_j \cup f_i^{j+1}$ forces a $t \in 2^{n_i}$. Define $g_i = f_i^{2^{n_i}}$ (to be used at the end of the proof of the Claim). By construction $f_i = f \supseteq g_i$ would satisfy (1).

For the condition (2) we will use an auxiliary definition. Let $s, s' \in 2^{n_i}$. A partial function f is a *witness* for $\langle s, s' \rangle$ if there is some $n > n_i$ such that

3. $f : [n_i, n) \to 2$ and

4. for any $k > n$, $g : [n, k) \to 2$, $t, t' \in 2^{<\omega}$ such that $s \cup f \cup g$ forces t and $s' \cup f \cup g$ forces t' then $t(j) = t'(j)$ for all $j \geq n$.

Claim 3.1.3. For every partial function $h : [n_i, k) \to 2$ and every $s, s' \in 2^{n_i}$ there is a witness $f \supseteq h$ of $\langle s, s' \rangle$.

Proof. Suppose for a contradiction that there exist $s, s \in 2^{n_i}$ and $h : [n_i, k) \to 2$ such that there is no f extending h which witness $\langle s, s' \rangle$. Then we can define sequences of partial functions $\langle h^j : j \in \omega \rangle$ (by iteratively defining $h^0 = h$ and extending h^j by using the hypothesis), $\langle t^j : j \in \omega \rangle$, $\langle t'^j : j \in \omega \rangle$ and an increasing sequence of integers $\langle l^j : j \in \omega \rangle$ such that $s \cup h^j$ forces t^j and $s' \cup h^j$ forces t'^j and $t^j(l^j) \neq t'^j(l^j)$. Then define $x = s \cup \bigcup \{h^j : j \in \omega\}$ and $x' = s' \cup \bigcup \{h^j : j \in \omega\}$. Then we have clearly $x =^* x'$. Yet $F(x) \neq^* F(x')$ which is a contradiction since F lifts φ. □

For the condition (2), we will apply the Subclaim repeatedly for each $\langle s, s' \rangle \in 2^{n_i} \times 2^{n_i}$. Enumerate $2^{n_i} \times 2^{n_i}$ as $\{\langle s^j, s'^j \rangle : j \in 2^{2n_i}\}$. Define by induction the $\langle g_i^j : j \in 2^{2n_i} \rangle$ and $\langle n_i^j :\in 2^{2n_i} \rangle$ as follows: Define $g_i^0 = g_i$ and $n_i^0 = n_i$. Let g_i^j and n_i^j be given. By the Subclaim we can find $n_i^{j+1} > n_i^j$ and g_i^{j+1} such that $g_i^{j+1} \supseteq g_i^j$ where $g_i^{j+1} : [n_i, n_i^{j+1}) \to 2$ and g_i^{j+1} is a witness of $\langle s^j, s'^j$. Let $f_{i+1} = g_i^{2^{2n_i}}$ and $n_{i+1} = n_i^{2^{2n_i}}$. This concludes the proof of the Claim. □

Now define $a^\epsilon = \bigcup \{[n_i, n_{i+1} : i = \epsilon \mod 3\}$ and $f^\epsilon = \bigcup \{f_i : i = \epsilon \mod 3\}$. Then consider the function $F^0(x) = F(x \cup a^1 \cup a^2) \setminus F(a^1 \cup a^2)$. Since φ is a homomorphism, $F^0(x) =^* F(x)$ for all $x \subseteq a^0$. Then we can find functions h_i such that

1. $h_i : \mathcal{P}([n_{3i}, n_{3i+1})) \to \mathcal{P}([n_{3i-1}, n_{3i+2}))$ for $i \in \omega$,

2. $F^0(x) = \bigcup \{h_i(x \cap [n_{3i}, n_{3i+1})) : i \in \omega\}$.

Claim 3.1.4. For all but finitely many $i \in \omega$ we have

$$h_i(u \cup v) = h_i(u) \cup h_i(v)$$

for all $u, v \in \mathcal{P}([n_{3i}, n_{3i+1}))$.

Proof. Assume for a contradiction that there is an infinite $A \subseteq \omega$ such that fr every $i \in A$ there are $u_i, v_i \subseteq [n_{3i}, n_{3i+1})$ such that $h_i(u_i \cup v_i) \neq h_i(u_i) \cup h_i(v_i)$.

Let $u = \bigcup\{u_i : i \in A\}$ and $v = \bigcup\{v_i : i \in A\}$. Then $F(u \cup v) \neq^* F(u) \cup F(v)$. This contradicts that φ is a homomorphism. □

Claim 3.1.5. h_i maps singletons to singletons for all but finitely many $i \in \omega$.

Proof. Assume for a contradiction that there is an infinite $A \subseteq \omega$ and $l_i \in [n_i, n_{i+1})$ for all $i \in A$ such that $h_i(\{l_i\})$ is not a singleton. Take $m_i \in h_i(\{l_i\})$ for all $i \in A$ and define $y = \{m_i : i \in A\}$. Then there is no x such that $F(x) =^* y$. This contradicts φ is surjective. □

Therefore we can define a function $h^0 : a^0 \to \omega$ by $h^0(l) = k$ if $l \in [n_{3i}, n_{3i+1})$ and $g_i(\{l\}) = k$. Then h^0 induces φ on $\mathcal{P}(a^0)/\mathrm{Fin}$. Similarly we can find h^1 and h^2 inducing φ respectively on $\mathcal{P}(a^1)/\mathrm{Fin}$ and $\mathcal{P}(a^2)/\mathrm{Fin}$. Combining these three functions and changing on a finite set would give a bijection $h : \omega \to \omega$ defined almost everywhere witnessing that φ is trivial.

To finish the proof we will consider the general case that F is defined on a dense G_δ set X. Then there are dense open sets U_n such that $X = \bigcap_n U_n$. Then construct inductively an increasing sequence $\langle m_k : k \in \omega\rangle$ and a sequence of finite functions $\langle f_k : k \in \omega\rangle$ such that $f_k : [m_k, m_{k+1}[\to 2$ and $N_{s \cup f_k} \subseteq U_k$ for every $s \in 2^{m_k}$. Then let $f^\epsilon = \bigcup\{f_k : k = \epsilon \mod 2\}$ and $a^\epsilon = \bigcup\{[m_k, m_{k+1}) : k = \epsilon mod 2\}$. Then for every $x \subseteq a^\epsilon$ we have $x \cup f^{1-\epsilon} \in X$. Therefore we can define $F^\epsilon : \mathcal{P}(a^\epsilon) \to 2$ by $F^\epsilon(x) = F(x \cup f^{1-\epsilon}) \setminus F(f^{1-\epsilon})$. As above we can find h^0 and h^1 inducing φ on $\mathcal{P}(a^0)/\mathrm{Fin}$ and $\mathcal{P}(a^1)/\mathrm{Fin}$. By combining them we obtain the desired result. □

An automorphism φ of $\mathcal{P}(\omega)/\mathrm{Fin}$ is called a *Borel automorphism* if there is a Borelian map $F : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ which lifts to φ.

Corollary 3.1.6. *Every Borel automorphism of* $\mathcal{P}(\omega)/\mathrm{Fin}$ *is trivial*

Proof. Let $\varphi \in \mathrm{Aut}(\mathcal{P}(\omega)/\mathrm{Fin})$ be a Borel automorphism. Then it is a well known fact that its graph $\Gamma(\varphi) \subseteq 2^\omega \times 2^\omega$ is Borel with countable sections. By Lusin-Novikov Theorem (see [Kec95, 18.10]), these sets can be uniformised by Borel sets. Thus there is a function $F : 2^\omega \to 2^\omega$ lifting φ and $\Gamma(F)$ is Borel. It is also well known (see [Kec95, 8.38]) that such F is continuous on a dense G_δ set $X \subseteq 2^\omega$. By Theorem 3.1.1 we achieve the desired result. □

3.2 Countably many Borel choice functions

Theorem 3.2.1 (Theorem 1.2 in [Vel93])**.** *Let* $\varphi \in \mathrm{Aut}(\mathcal{P}(\omega)/\mathrm{Fin})$ *and* $F : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ *such that* $\varphi(a) =^* F(a)$ *for every* $a \in \mathcal{P}(\omega)$. *Suppose there exist Borel functions* $F_n : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ *for* $n \in \omega$ *satisfying for every* $a \in \mathcal{P}(\omega)$ *there exists* n *such that* $F(a) =^* F_n(a)$. *Then* φ *is trivial.*

Define the *trivial ideal*

$$\mathcal{J} = \{a \in \mathcal{P}(\omega) : \varphi \restriction a \text{ is trivial}\}.$$

Let $(\mathbb{P}, \leq)$ be a poset. A subset $X \subseteq \mathbb{P}$ is called *cofinal* if for every $p \in \mathbb{P}$ there is some $x \in X$ such that $p \leq x$. Proof of the theorem requires the following lemmas.

Lemma 3.2.2. *$\mathcal{J}$ is not a maximal non principal ideal.*

Proof. Suppose for a contradiction that $\mathcal{J}$ is a maximal nonprincipal ideal. Fix a dense G_δ subset $X \subseteq \mathcal{P}(\omega)$ such that $F_n \restriction X$ is continuous for all n.

Claim 3.2.3. There exist a coloring of $\omega = a_0 \cup a_1$ and $t_\epsilon \subseteq a_\epsilon$ for $\epsilon \in 2$ such that if $x \subseteq a_\epsilon$ then $x \cup t_{1-\epsilon} \in X$.

Proof. Fix a decreasing sequence $\langle U_n : n \in \omega\rangle$ of dense open sets such that $X = \bigcap_{n\in\omega} U_n$. Build inductively an increasing sequence of integers $\langle n_i : i \in \omega\rangle$ with $n_0 = 0$ and a sequence of sets $\langle s_i : i \in \omega\rangle$ such that $s_i \subseteq [n_i, n_{i+1})$ and for all $x \in \mathcal{P}(\omega)$ if $x \cap [n_i, n_{i+1}) = s_i$ then $x \in U_i$. Define $a_\epsilon = \bigcup\{[n_i, n_{i+1}) : i = \epsilon \mod 2\}$ and $t_\epsilon = \bigcup\{s_i : i = \epsilon \mod 2\}$ for $\epsilon \in 2$.

Fix an $\epsilon \in 2$. Pick $x \subseteq a_\epsilon$. Then it is clear that $x \cup t_{1-\epsilon} \in U_n$ for all $n \in \omega$ since $(x \cup t_{1-\epsilon}) \cap [n_i, n_{i+1}) = s_i$ for all $i = 1 - \epsilon \mod 2$. Therefore this construction works. □

Now we return to the proof of the Lemma 3.2.2 and fix a coloring as in the Claim 3.2.3. Suppose that φ is nontrivial on a_0 and define a function

$$G_n(x) = F_n(x \cup t_1) \cap F(a_0)$$

for each $n \in \omega$. Since each F_n is continuous on X, G_n are also continuous on $\mathcal{P}(a_0)$ and for every $x \subseteq a_0$ there exists $n \in \omega$ such that $G_n(x) =^* F(x)$. Let $\mathcal{I}$ be the restriction of $\mathcal{J}$ to $\mathcal{P}(a_0)$. Then for every $a \in \mathcal{I}$ pick a one-to-one function $e_a : a \to \omega$ inducing φ on a and define the function $E_a : \mathcal{P}(a) \to \mathcal{P}(\omega)$ by $E_a(x) = e_a[x]$. Then E_a is continuous and $E_a(x) =^* F(x)$ for all $x \subseteq a$.

Now fix an $a \in \mathcal{I}$ and define the sets

$$D^a_{n,m} = \{x : E_a(x) \setminus m = G_n \setminus m\}$$

where $n, m \in \omega$. Observe that $\mathcal{P}(a) = \bigcup_{n,m\in\omega} D^a_{n,m}$. Then there are some $n, m \in \omega$ such that $D^a_{n,m}$ is dense in some clopen $U \subseteq \mathcal{P}(a)$ since countable union of nowhere dense sets would be nowhere dense.

Now define the quintuples $\langle i, l, m, s, t\rangle$ such that $i, l, m, \in \omega$, $t \subseteq i$ and $s : \mathsf{dom} s \to \omega$ is a function with $\mathsf{dom} s \in \omega$. Let $\{\langle i_n, l_n, m_n, s_n, t_n\rangle : n \in \omega\}$ be an enumeration of all these quintuples. Define a function H_n on $\mathcal{P}(a_0)$ by

$$H_n(x) = (G_{l_n}((x \setminus i_n) \cup t_n) \setminus m_n) \cup s_n[x \cap \mathsf{dom}(s_n)].$$

$\{H_n : n \in \omega\}$ is a family of continuous functions on $\mathcal{P}(a_0)$ such that for every $a \in \mathcal{I}$ there exists $n \in \omega$ such that $H_n(x) = e_a[x]$ for all $x \subseteq a$. Then define the sequence

$$\mathcal{I}_n = \{a \in \mathcal{I} : H_n(x) = e_a[x] : \text{ for all } x \subseteq a\}.$$

Now if we suppose that one of $\mathcal{I}_n$ is cofinal in $(\mathcal{I}, \subseteq^*)$ we can define $e = \bigcup\{e_a : a \in \mathcal{I}_n\}$ since for any $a, b \in \mathcal{I}_n$ we have e_a is equal to e_b on $a \cap b$. Therefore e induces φ on every $a \in \mathcal{I}$. Therefore e induces φ on a_0 which is a contradiction.

So none of $\mathcal{I}_n$ is cofinal in $\mathcal{I}$. Then we can find a decomposition $a_0 = \bigcup_{n\in\omega} b_n$ such that $b_n \in \mathcal{I}$. Therefore there is no $b \in \mathcal{I}$ with $b_n \subseteq^* b$ for all n. Let $\mathcal{A}$ be the set of all $b \subseteq a_0$ which are almost disjoint from all the b_n. Then $\mathcal{A} \subseteq \mathcal{I}$ is a σ-directed sub ideal by A.3.2. Let $\mathcal{A}_n = \mathcal{A} \cap \mathcal{I}_n$. Then there exists $n \in \omega$ such that $\mathcal{A}_n$ is cofinal in $\mathcal{A}$. Now define $e = \bigcup\{e_a : a \in \mathcal{A}_n\}$ as above. This then the following claim implies that φ is trivial on a_0.

Claim 3.2.4. There exists $k \in \omega$ such that e induces φ on $a_0 \setminus \bigcup_{i<k} b_i$.

Proof. Suppose for a contradiction that $T = \{n \in \omega : e \restriction b_n \text{ is not trivial}\}$ is infinite. Then for each $m \in T$ pick an infinite $c_m \subseteq b_m$ such that $e[c_m] \cap F[c_m] \in \text{Fin}$. Moreover for any $m, k \in T$, we can find c_m and c_k satisfying $e[c_m] \cap F(c_k) \in \text{Fin}$ by shrinking c_m, c_k. Therefore we can find $d \subseteq \omega$ such that $F(c_m) \subseteq^* d$ for all $m \in T$. Since F is surjective, there is $c \subseteq \omega$ with $F(c) =^* d$. Therefore $c_m \subseteq^* c$ for all $m \in T$. Then we can pick $i_m \in c_m \cap c$ such that $e(i_m) \notin F(c)$. Define $b = \{i_m : m \in T\}$. Then clearly $b \in \mathcal{A}$. This implies that $F(b) =^* e[b]$. Also since $b \subseteq c$ we have $F(b) \subseteq^* F(c)$. Yet $e[b] \cap F(c) \in \text{Fin}$ by definition of b. Contradiction. □

So T should be finite. Then e induces $\varphi \restriction a$ for all a in the ideal generated by $\mathcal{A}$ and $\{b_m : m \notin T\}$. Because this ideal is dense in $\mathcal{P}(u)$ where $u = a \setminus \bigcup_{m\in T} b_m$. □

Proof of the Theorem 3.2.1. Now we return to the proof of the theorem. Assume that φ is nontrivial and build inductively disjoint sets a_n and x_n such that for every $n \in \omega$

- $x_n \subseteq a_n$,
- φ is nontrivial on $\omega \setminus \bigcup_{i\le n} a_i$,
- for every $\omega \setminus \bigcup_{i\le n} a_i$ we have

$$F_n\left(\bigcup_{i\le n} x_i \cup x\right) \cap F(a_n) \neq^* F(x_n).$$

Suppose $\langle a_i : i \in \omega \rangle$ and $\langle x_i : i \in \omega \rangle$ are constructed. Let $c_n = \omega \setminus \bigcup_{i<n} a_i$ and $z_n = \bigcup_{i<n} x_i$. By the Lemma 3.2.2 there is a decomposition $c_n = d_n \cup e_n$ such that φ is nontrivial on both components. For $y \subseteq d_n$ define

$$B_n(y) = \{x \subseteq e_n : F_n(z_n \cup y \cup x) \cap F(d_n) =^* F(y)\}.$$

By definition, $B_n(y) \subseteq \mathcal{P}(e_n)$ is a Borel set.

Claim 3.2.5. There exists $y \subseteq d_n$ such that $B_n(y)$ is not comeager.

Proof. Suppose for a contradiction that $B_n(y)$ is comeager for all $y \subseteq d_n$. Let $\Gamma(\varphi \restriction d_n)$ be the graph of $\varphi \restriction d_n$. For any $\langle y, u \rangle \in \Gamma(\varphi \restriction d_n)$ we have $\{x \subseteq e_n : F_n(z_n \cup y \cup x) \cap F(d_n) =^* u\}$ is comeager. Therefore $\Gamma(\varphi \restriction d_n)$ is analytic can be uniformised on a comeager set by a continuous function. Thus by the Theorem 3.1.1 we have φ is trivial on d_n. Yet this contradicts that φ is nontrivial on d_n. ☐

Now we will construct the sequences $\langle a_n : n \in \omega \rangle$ and $\langle x_n : n \in \omega \rangle$ inductively. Fix some $y \subseteq d_n$ and a standard clopen set $N_s \subseteq \mathcal{P}(e_n)$ such that $B_n(y)$ is meager in N_s. Let $u_0 = s^{-1}(0)$ and $u_1 = s^{-1}(1)$ and $u = u_0 \cup u_1$. Find a decomposition $e_n \setminus \mathsf{dom} s = e_n^0 \cup e_n^1$ and subsets $t_\epsilon \subseteq e_n^\epsilon$ where $\epsilon \in 2$ as done in the Claim 3.2.3 satisfying $u \cup x \cup t_{1-\epsilon} \notin B_n(y)$ for all $x \subseteq e_n^\epsilon$. Then there is some $\epsilon \in 2$ such that φ is nontrivial on e_n^ϵ. Assume without loss of generality that this is the case for $\epsilon = 0$. Set $a_n = d_n \cup u_0 \cup e_n^1$ and $x_n = y \cup u_1 \cup t_1$.

Let $x = \bigcup_{n \in \omega} x_n$. For every $n \in \omega$ we have $F_n(x) \cap F(a_n) \neq^* F(x_n)$. Yet by the hypotheses of the theorem there is some $n \in \omega$ such that $F_n(x) =^* F(x)$. Contradiction. ☐

Chapter 4

Embedding $\mathcal{P}(\omega)/\mathrm{Fin}$ into $\mathcal{C}(X)/\mathcal{K}(X)$

In the first two section of this chapter, we prove some lemmas which are the main components of the proof of the main theorem of this report. These lemmas are generalisations of the analogues from [Vel93] which are given in [FM12] without poofs. Then by using these lemmas, we prove the main theorem in the third section.

Fix for the rest of the chapter an injective homomorphism $\varphi : \mathcal{P}(\omega)/\mathrm{Fin} \to \mathcal{C}(X)/\mathcal{K}(X)$, a corresponding function $F : \mathcal{P}(\omega) \to \mathcal{C}(X)$ and define the ideal of trivial sets

$$\mathcal{J} = \{a \in \mathcal{P}(\omega) : \varphi \restriction a \text{ is trivial}\}.$$

4.1 P-ideals

In this section we will study the triviality with respect to P-ideals.

Definition 4.1.1. An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called a *P-ideal* if for each countable sequence $\langle A_n \in \mathcal{I} : n \in \omega \rangle$ there is an $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for all $n \in \omega$.

Lemma 4.1.2 (Lemma 2.4 in [Vel93])**.** *Assume* OCA *and* $\mathsf{MA}_{\aleph_1}$*. If $\mathcal{J}$ is a dense P-ideal then φ is trivial.*

Proof. For every $a \in \mathcal{J}$ fix a $Z_a \in \mathcal{C}(X)$ and a compact-to-one function $e_a : Z_a \to a$ such that $\varphi([a]) = [Z_a]$ and $\varphi([b]) = [e^{-1}(b)]$ for all $b \subseteq a$. Then define $f_a : \omega \to \mathcal{C}(X)$ by $f_a(n) = e^{-1}(\{n\})$.

Define the partition $[\mathcal{J}]^2 = M_0 \cup M_1$ as $\{a, b\} \in M_0$ if and only if there is some $n \in a \cap b$ such that $f_a(n) \neq f_b(n)$. Since this condition is existential, M_0 is open in the topology obtained by identifying $a \in \mathcal{J}$ with $(a, f_a) \in \mathcal{P}(\omega) \times \mathcal{C}(X)^\omega$.

Claim 4.1.3. There is no uncountable M_0-homogeneous subset $H \subseteq \mathcal{J}$.

Proof. Suppose for a contradiction that these is an M_0-homogeneous H with $|H| = \aleph_1$. Since $\mathcal{J}$ is a P-ideal, there exists $\overline{H} \subseteq \mathcal{J}$ such that for every $a \in H$ there is a $b \in \overline{H}$ with $a \subseteq^* b$ and order type of $(\overline{H}, \subseteq^*)$ is ω_1. OCA implies that $\overline{H}$ has an uncountable subset which is either M_0-homogeneous or M_1-homogeneous. By shrinking, we shall suppose that the subset is $\overline{H}$.

Suppose for a contradiction that $\overline{H}$ is M_1-homogeneous. Define $\bar{a} = \bigcup \overline{H}$ and $\bar{f} = \bigcup_{a \in \overline{H}} f_a$ with $\bar{f} : \bar{a} \to \mathcal{C}(X)$. Then clearly $a \subseteq^* \bar{a}$ and $f_{\bar{a}} \restriction (\bar{a} \cap a) =^* f_a \restriction (\bar{a} \cap a)$ for all $a \in H$. Choose n such that for uncountably many $a \in H$ we have $a \setminus n \subseteq \bar{a}$ and $f_{\bar{a}} \restriction (a \setminus n) = f_a \restriction (a \setminus n)$. Take such $a, b \in H$ with $f_a \restriction n = f_b \restriction n$. Then $\{a, b\} \in M_1$, which is contradictory. Therefore $\overline{H}$ is M_0-homogeneous.

Let $\langle K_n : n \in \omega \rangle$ be an increasing compact cover of X which exists the Proposition 2.0.2. Define a poset $\mathbb{P}$ by $p \in \mathbb{P}$ if and only if $p = (A_p, m_p, H_p)$ where $m_p \in \omega$, $A_p \in \mathcal{C}(K_{m_p})$ and $H_p \in [\overline{H}]^{<\omega}$ satisfying for any $a, b \in H_p$ there is an $n \in a \cap b$ such that either

$$f_a(n) \cap A_p = \emptyset \text{ and } f_b(n) \cap A_p \neq \emptyset$$

or

$$f_b(n) \cap A_p = \emptyset \text{ and } f_a(n) \cap A_p \neq \emptyset$$

equipped with the order $p \leq q$ if and only if $m_p \geq m_q$, $A_p \cap K_{m_q} = A_q$ and $H_p \supseteq H_q$.

Claim 4.1.4. $\mathbb{P}$ is ccc.

Proof. Let $X \subseteq \mathbb{P}$ be uncountable. By the pigeonhole principle we may assume that there is a fix $m \in \omega$ and $A \in \mathcal{C}(K_m)$ such that $m_p = m$ and $A_p = A$ for all $p \in X$. Moreover we may assume that $|H_p|$ is the same for all $p \in X$.

Let a_p be the $\subseteq^*$-minimal element of H_p for each $p \in X$. Find n_p such that for all $a \in H_p$ satisfying $f_{a_p} \restriction (a_p \setminus n_p) \subseteq f_a$ and $e_{a_p}(K_m) \subseteq n_p$.

Similarly we may assume that for some fixed n we have $n_p = n$ for all $p \in X$. Find $p, q \in X$ such that $f_{a_p} \restriction n = f_{a_q} \restriction n$. As we have $\{a_p, a_q\} \in M_0$, there is some $k \in a_p \cap a_q$ such that $f_{a_p}(k) \neq f_{a_q}(k)$. Therefore $k \geq n$ and $f_{a_p}(k) \cap K_m = f_{a_q}(k) \cap K_m = \emptyset$. Then $f_{a_p}(k) \setminus f_{a_q}(k)$ or $f_{a_p}(k) \setminus f_{a_q}(k)$ is non empty. Call the non-empty one B. Define $r \in \mathbb{P}$ such that $A_r = A \cup B$, $H_r = H_p \cup H_q$ and pick m_r large enough to satisfy $A_r \subseteq K_{m_r}$. Then clearly $r \leq p, q$. □

By $\mathsf{MA}_{\aleph_1}$, there is a set $A \in \mathcal{C}(X)$ and an uncountable $H^* \subseteq \overline{H}$ such that for all distinct $a, b \in H^*$, there is some $n \in a \cap b$ such that either $f_a(n) \cap A_p = \emptyset$ and $f_b(n) \cap A_p \neq \emptyset$ or $f_b(n) \cap A_p = \emptyset$ and $f_a(n) \cap A_p \neq \emptyset$. Let $x \subseteq \omega$ with $F(x) = A$. Then for all $a \in H^*$ we have $e^{-1}(x \cap a) \triangle (A \cap F(a))$ is compact. So there are k_a and m_a such that $e_a^{-1}(x \cap a \setminus k_a) = (A \cup F(a)) \setminus K_{m_a}$ and

$e_a^{-1}(a \setminus k_a) = F(a) \setminus K_{m_a}$. Then for all $n \in a \setminus k_a$, if $n \in x$ then $f_a(n) \subseteq A$ and if $n \notin x$ then then $f_a(n) \cap A = \emptyset$. Pick distinct $a, b \in H^*$ satisfying $k_a = k_b = k$ and $f_a \restriction k = f_b \restriction k$. Then we have $f_a(n) \cap A = \emptyset$ if and only if $f_b(n) \cap A = \emptyset$ for all $n \in a \cap b$. Yet this contradicts the assumptions on A. □

Now by OCA, there is a decomposition $\mathcal{J} = \bigcup_{n\in\omega} \mathcal{J}_n$ where $\mathcal{J}_n$ is M_1-homogeneous for each n. As $\mathcal{J}$ is a P-ideal, there is some $n \in \omega$ such that $\mathcal{J}_n$ is cofinal in $(\mathcal{J}, \subseteq^*)$. Let T_n be such then define $f = \bigcup\{f_a : a \in \mathcal{J}_n\}$. Set $e(x) = n$ if and only if $x \in f(n)$. Then since $\mathcal{J}$ is dense and $\mathcal{J}_n \subseteq \mathcal{J}$ is cofinal, $a \mapsto e^{-1}(a)$ witnesses that φ is trivial. □

Definition 4.1.5. An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ containing Fin is called $P_{\aleph_1}$*-ideal* if for every family $\mathcal{F} \subseteq \mathcal{I}$ of size $\aleph_1$ there is some $A \in \mathcal{I}$ such that $B \subseteq^* A$ for all $B \in \mathcal{F}$.

Remark 4.1.6. OCA implies that if $\mathcal{I}$ is a $P_{\aleph_1}$-ideal and $\{f_a : a \in \mathcal{I}\}$ is a family of functions such that $f_a : a \to \omega$ and $f_b \restriction a =^* f_a$ whenever $a, b \in \mathcal{I}$ and $a \subseteq b$ then there exists $f : \omega \to \omega$ such that $f \restriction a = f_a$ for all $a \in \mathcal{I}$ (see [Far00, Chapter 2.2]).

Lemma 4.1.7 (Lemma 2.5 in [Vel93])**.** *Assume $\mathfrak{b} > \aleph_1$. If $\mathcal{J}$ is not a dense P-ideal then there is an uncountable almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A} \cap \mathcal{J} = \emptyset$.*

Proof. Suppose first that $\mathcal{J}$ is not dense. Then there is some infinite $A \in \mathcal{P}(\omega)$ such that there is no $B \in \mathcal{J}$ satisfying $B \subseteq A$. In other words $\mathcal{P}(A) \cap \mathcal{J} = \emptyset$. Then we can find some uncountable almost disjoint $\mathcal{A} \subseteq \mathcal{P}(A)$.

Now suppose that $\mathcal{J}$ is dense but not a P-ideal. Then there is a sequence $\langle A_n : n \in \omega\rangle$ such that there is no $A \in \mathcal{J}$ satisfying $A_n \subseteq^* A$. We can suppose without loss of generality that $\bigcup_{n\in\omega} A_n = \omega$ and A_n are pairwise disjoint. For $f \in \omega^\omega$ define $B_f = \bigcup\{A_n \cap f(n) : n \in \omega\}$.

Claim 4.1.8. There exists $f \in \omega^\omega$ with φ is nontrivial on B_f.

Proof. Suppose for a contradiction that $B_f \in \mathcal{J}$ for all $f \in \omega^\omega$. Therefore there exists $e_f : B_f \to \omega$ witnessing the triviality of φ on B_f. Define $\mathcal{I}$ as all B_f which are almost disjoint from A_n for all n. Since $\mathfrak{b} > \aleph_1$, we have $\mathcal{I}$ is $P_{\aleph_1}$-subideal of $\mathcal{J}$. Moreover from the Remark 4.1.6, there exists $e : X \to \omega$ such that $e \restriction B =^* e_B$ for every $B \in \mathcal{I}$. Note that we abuse the notation for simplicity by writing $e \restriction B$. Here we mean clearly $e^{-1} \restriction B$.

Claim 4.1.9. There exists $k \in \omega$ such that e induces φ on $\omega \setminus \bigcup_{i<k} A_i$.

Proof. It is sufficient to show that $S = \{n \in \omega : e \restriction A_n$ does not induce $\varphi \restriction A_n\}$ is finite.

Suppose for a contradiction that S is infinite. Then for all $n \in S$, choose an infinite $C_n \subseteq A_n$ such that $e^{-1}(C_n) \cap F(C_n) \in \mathcal{K}(X)$. By shrinking C_n we can arrange that $e^{-1}(C_n) \cap F(C_m) \in \mathcal{K}(X)$ for every $n, m \in S$. Then find some $U \in \mathcal{C}(X)$ such that $F(C_n) \subseteq^* U$ and $e^{-1}(C_n) \cap U \in \mathcal{K}(X)$ for all $n \in S$. Let $F(C) =^* U$. Pick some $i_n \in C_n \cap C$ such that $e^{-1}(\{i_n\}) \not\subseteq^* F(C_n)$. Observe that $B = \{i_n : n \in S\}$ is trivial. So $e^{-1}(B) =^* F(B)$. Since $B \subseteq C$ we have $F(B) \subseteq^* F(C)$. Yet $e^{-1}(B) \cap F(C) \in \mathcal{K}(X)$. Contradiction. □

Yet the Subclaim contradicts with the nontriviality of φ. This completes the proof of the Claim. □

For any $f \in \omega^\omega$ we can find some $g \in \omega^\omega$ with $f \leq^* g$ and $B_f \setminus B_g$ is nontrivial by applying the same reasoning in the Claim for above $A_n \cap f(n)$. Since we assumed $\mathfrak{b} > \omega_1$, we can inductively construct an $\leq^*$-increasing sequence $\langle f_\alpha : \alpha \in \omega_1 \rangle$ such that $B_{f_{\alpha+1}} \setminus B_{f_\alpha}$ are nontrivial. Therefore $\mathcal{A} = \{B_{f_{\alpha+1}} \setminus B_{f_\alpha} : \alpha \in \omega_1\}$ is the desired family. □

4.2 Tree-like families

A tree-like family is a family of infinite subsets of ω whose elements correspond to the infinite branches of the tree $2^{<\omega}$. In this section we will see that under $\mathsf{MA}_{\aleph_1}$, any uncountable almost disjoint family contains an uncountable subfamily which can be divided into two tree-like families.

Definition 4.2.1. An almost disjoint family $\mathcal{A}$ is *tree-like* if there is a tree T on ω and an injection $t : \omega \to 2^{<\omega}$ such that for each $a \in \mathcal{A}$ and each $m, n \in a$, either $t(m) \subseteq t(n)$ or $t(n) \subseteq t(m)$.

Lemma 4.2.2 (Lemma 2.3 in [Vel93])**.** *Assume $\mathsf{MA}_{\aleph_1}$. Then for every uncountable almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ admits an uncountable $\mathcal{B} \subseteq \mathcal{A}$ and partitions $b = b_0 \sqcup b_1$ for $b \in \mathcal{B}$ such that $\mathcal{B}_i = \{b_i : b \in \mathcal{B}\}$ is tree-like for $i \in 2$.*

Proof. Following [Vel93], define the poset $\mathbb{P}$ such that $p \in \mathbb{P}$ if and only if $p = (e^0_p, e^1_p, n_p, A_p, D_p)$ where

1. $n_p \in \omega$,
2. $e^i_p : n_p \to 2^{<\omega}$ is into,
3. $A_p \subseteq \mathcal{A}$ is finite such that for any two different $a, b \in A_p$ we have $a \cap b \subseteq n_p$,
4. $D_p = \{f^a_p : a \in A_p\}$ where $f^a_p : a \cap n_p \to 2$ for every $a \in A$,
5. for every $k, l < n_p$, if there exists $a \in A_p$ such that $k, l \in a$ and $f^a_p(k) = f^a_p(l) = i$ for some $i \in 2$ then $e^i_p(k) \subseteq e^i_p(l)$ or $e^i_p(l) \subseteq e^i_p(k)$,

with the order $p \leq q$ if and only if $n_q \leq n_p$, $e^i_q \subseteq e^i_p$ for $i \in 2$, $A_q \subseteq A_p$ and $f^a_q \subseteq f^a_p$ for every $a \in A_q$.

Claim 4.2.3. $\mathbb{P}$ is ccc.

Proof. Let $S \subseteq \mathbb{P}$ be a subset of size $\aleph_1$. By the pigeonhole principle we may suppose that there exist $n \in \omega$ and $e^i : n \to 2^{<\omega}$ such that $n_p = n$ and $e^i_p = e^i$ for all $p \in S$ and for all $i \in 2$. Moreover by the Δ-system lemma we may assume that $\{A_p : p \in S\}$ is a Δ-system with root A. Assume therefore that for each $a \in A$, there is $f^a : a \cap n \to 2$ such that $f^a_p = f^a$ for all $p \in S$.

Pick two different elements $p, q \in S$. We will construct $r \leq p, q$. Let $A_r = A_p \cup A_q$ and pick $n_r \geq n$ sufficiently large such that $a \cap b \subseteq n_r$ for any distinct $a, b \in A_r$. Suppose that $a \in A_r$. Define $f^a_r : a \cap n_r \to 2$ such that

$$f^a_r(k) = \begin{cases} f^a_p(k) & \text{if } a \in A_p \text{ and } k < n \\ 0 & \text{if } a \in A_p \text{ and } n \leq k \leq n_r \\ f^a_q(k) & \text{if } a \in A_q \setminus A_p \text{ and } k < n \\ 1 & \text{if } a \in A_q \setminus A_p \text{ and } n \leq k \leq n_r \end{cases}$$

So $D_r = \{f^a_r : a \in A_r\}$.

Lastly we define $e^i_r : n_r \to 2^{<\omega}$ as follows: $e^i_r \restriction n = e^i$. We have already that $(e^i)''(a \cap n)$ is a chain in $2^{<\omega}$ moreover $\{a \setminus n : a \in A_p\}$ and $\{a \setminus n : a \in A_q \setminus A_p\}$ families of disjoint sets. So we can define $e^0_r \restriction [n, n_r[$ to $(e^0_r)''(a \cap n_r)$ be a chain for every $a \in A_p$ and $e^1_r \restriction [n, n_r[$ to $(e^1_r)''(a \cap n_r)$ be a chain for every $a \in A_q \setminus A_p$ respecting the rules (2) and (5). From the construction it is clear that $r \in \mathbb{P}$ and $r \leq p, q$. □

Now assume without loss of generality that $|\mathcal{A}| = \aleph_1$ and define $D_{a,n} = \{p \in \mathbb{P} : a \in A_p \text{ and } n_p \geq n\}$ where $a \in \mathcal{A}$ and $n \in \omega$.

Claim 4.2.4. $D_{a,n} \subseteq \mathbb{P}$ is dense for all $a \in \mathcal{A}$ and $n \in \omega$

Proof. Fix $a \in \mathcal{A}$ and $n \in \omega$. Pick $p \in \mathbb{P}$ where $p = (e^0_p, e^1_p, n_p, A_p, D_p)$. Then we shall find $q \in D_{a,n}$ as follows: Let $A_q = A_p \cup \{a\}$. Then let $n_q \geq \max n_p, n$ be the smallest $k \supseteq a \cap b$ for all $b \in A_p$. Then we can find $f^x_q \supseteq f^x_p$ and $e^i_q \supseteq e^i_p$. Observe that $q \in D_{a,n}$ and $q \leq p$ as desired. □

By the $\mathsf{MA}_{\aleph_1}$, there is a $\{D_{a,n} : a \in \mathcal{A}, n \in \omega\}$-generic filter G. Then for all $a \in \mathcal{A}$ there is a total function $f^a : a \to 2 = \bigcup\{f^a_p : p \in G, a \in A_p\}$. Define $a_0 = (f^a)^{-1}(0)$ and $a_1 = (f^1)^{-1}(1)$ as $a = a_o \cup a_1$. Moreover $e^i = \bigcup\{e^i_p : p \in G\}$ witness that $\{a_i : a \in \mathcal{A}\}$ is tree-like. □

Lemma 4.2.5 (Lemma 3.5 of [FM12])**.** *Assume* OCA*. Let* $\mathcal{A}$ *be an uncountable, tree-like, almost disjoint family of subsets of* ω*. Then* $\mathcal{J} \setminus \mathcal{A}$ *is countable.*

Proof. Let $t : \omega \to 2^{<\omega}$ be an injection witnessing that $\mathcal{A}$ is tree-like and X be the set of all pairs $\langle a, b, \rangle$ of subsets of ω such that there exists $c \in \mathcal{A}$ with $b \subseteq a \subseteq c$. Define the coloring $[X]^2 = M_0 \cup M_1$ by $\{\langle a, b\rangle, \langle \bar{a}, \bar{b}\rangle\} \in M_0$ if and only if

1. $t[a] \neq t[\bar{a}]$,
2. $a \cap \bar{b} = \bar{a} \cap b$ and
3. $F(a) \cap F(\bar{b}) \neq F(\bar{a}) \cap F(b)$.

Then M_0 is open in the product of the separable metric topology on X obtained by identifying $\langle a, b\rangle$ with $\langle a, b, F(a), F(b)\rangle$.

Claim 4.2.6. There are no uncountable M_0-homogeneous subsets of X.

Proof. Suppose for a contradiction that there is such $Y \subseteq X$. Define $d = \bigcup\{b : \langle a, b\rangle \in Y \text{ for some } a\}$. Pick $\langle a, b, \rangle \in Y$. By the second condition $d \cap a = b$ and so $F(d) \cap F(a) =^* F(b)$. By pigeonhole principle we can find an uncountable $Z \subseteq Y$ and $n \in \omega$ such that $(F(d) \cap F(a)) \triangle F(b) \subseteq K_n$ and $F(b) \setminus K_n \subseteq F(a)$ for all $\langle a, b\rangle \in Z$. Then there are distinct $\langle a, b\rangle$ and $\langle \bar{a}, \bar{b}\rangle$ in Z such that $F(a) \cap K_n = F(\bar{a}) \cap K_n$ and $F(b) \cap K_n = F(\bar{b}) \cap K_n$. Therefore we have $F(a) \cap F(\bar{b}) = F(\bar{a}) \cap F(b)$. This contradicts $\{\langle a, b\rangle, \langle \bar{a}, \bar{b}\rangle\} \in M_0$. □

Therefore OCA implies that there is a countable decomposition $X = \bigcup_{n\in\omega} X_n$ where X_n are M_1-homogeneous. Fix a countable dense subset $D_n \subseteq X_n$ in the sense of the product topology. For each $\langle a, b\rangle \in X$ pick $\sigma(a) \in \mathcal{A}$ such that $b \subseteq a \subseteq \sigma(a)$ and define $\mathcal{B} = \{\sigma(a) : \langle a, b\rangle \in D_n \text{ for some } n \in \omega\}$.

Now we show that φ is trivial on every $c \in \mathcal{A} \setminus \mathcal{B}$. Fix any such c and decompose it into two disjoint sets $c = c_0 \cup c_1$ such that foe every $\epsilon \in 2$, $n \in \omega$ and $\langle a, b\rangle \in X_n$ if $a \subseteq c_\epsilon$ then for every $m \in \omega$ there exists $\langle \bar{a}, \bar{b}\rangle \in D_n$ such that:

1. $a \cap \bar{b} = \bar{a} \cap b$,
2. $a \cap m = \bar{a} \cap m$ and $b \cap m = \bar{b} \cap m$,
3. $F(a) \cap K_m = F(\bar{a}) \cap K_m$ and $F(b) \cap K_m = F(\bar{b}) \cap K_m$.

The decomposition is done as follows: First construct an increasing sequence $\langle n_i : i \in \omega\rangle$ by induction. Let $n_0 = 0$. Suppose $\langle n_i : i \leq k\rangle$ is defined. Then choose n_{k+1} sufficiently large such that for every $x, y \subseteq n_k$ and every $i \leq k$ if there exists $\langle a, b\rangle \in X_i$ such that $a \cap n_k = x$, $b \cap n_k = y$, $F(x) \subseteq K_{n_k}$ and $F(y) \subseteq K_{n_k}$ then there exists $\langle a, b\rangle \in D_i$ satisfying the same properties and $a \cap c \subseteq n_{k+1}$. Since a is almost disjoint from c we can find such b satisfying $\langle a, b\rangle \in D_i$.

Now define $c_0 = \bigcup\{c \cap [n_k, n_{k+1}) : k = 0 \mod 2\}$ and $c_1 = c \setminus c_0$. Moreover define the sequence of functions $F_n : \mathcal{P}(c_0) \to \mathcal{C}(X)$ by $F_n(b) = \bigcup\{F(c_0) \cap F(\bar{b}) : \langle \bar{a}, \bar{b}\rangle \in D_n \text{ and } \bar{a} \cap b = c_0 \cap \bar{b}\}$.

By definition F_n are Borel functions. Let $\langle c_0, b\rangle \in X_n$. Then $F_n(b) =^* F(b)$. By the Theorem 3.2.1, φ is trivial on c_0. By defining an analogue sequence of functions we show that φ is trivial on c_1. Therefore we have φ is trivial on c. □

4.3 Embedding Theorem

In this section we will show the main result of this report: every embedding of $\mathcal{P}(\omega)$ in $\mathcal{C}(X)/\mathcal{K}(X)$ is trivial under $\mathsf{OCA} + \mathsf{MA}_{\aleph_1}$. We will prove this by showing that $\mathcal{J}$ is a dense P-ideal. By virtue of the lemmas of the previous sections we will prove that $\mathcal{J}$ is a P-ideal. We will quote some consequences of the independent works of Jalali-Naini and Talagrand ([JN76] , [Tal80]) which characterise comeager subsets of certain compact spaces and as a result show that $\mathcal{J}$ is dense. Recall that $Y \subseteq X$ is *meager*, if Y is a countable union of nowhere-dense sets and *comeager* if $X \setminus Y$ is meager.

Following theorems are from the paper [MV21]. These are due to Jalali-Naini and Talagrand independently. Their proofs can be found in [Far00, Chapter 3.10].

Theorem 4.3.1 (Theorem 2.1 of [MV21]). *Let Y_n be finite sets for $n \in \omega$. A set $G \subseteq \prod Y_n$ is comeager if and only if there is a partition $\langle E_i : i \in \omega \rangle$ of ω into finite intervals and a sequence $t_i \in \prod_{n \in E_i} Y_n$ such that $y \in G$ whenever $\{i : y \restriction (\prod_{n \in E_i}) = t_i\}$ is infinite.* □

A set $H \subseteq \mathcal{P}(\omega)$ is called *hereditary* if whenever $b \in H$ and $a \subseteq b$, we have $a \in H$. Any ideal of $\mathcal{P}(\omega)$ is hereditary by definition. We have the following theorem as a corollary of Theorem 4.3.1.

Theorem 4.3.2 (Proposition 2.4 of [MV21]). *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal containing* Fin. *Then the following are equivalent:*

1. *$\mathcal{I}$ has the Baire property;*
2. *$\mathcal{I}$ is meager;*
3. *there is a partition $\langle E_i : i \in \omega \rangle$ of ω into finite intervals such that for any infinite set L, $\bigcup_{n \in L} E_n$ is not in $\mathcal{I}$.* □

Corollary 4.3.3 (Corollary 3.10.2 of [Far00]). *A subset $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is comeager if and only if there is a sequence $0 = n_0 < n_1 < \cdots$ of natural numbers and $s_i \subseteq [n_i, n_{i+1})$ such that $\mathcal{I}$ includes the set $\{a \subseteq \omega : a \cap [n_i, n_{i+1}) = s_i$ for infinitely many $i\}$.* □

Theorem 4.3.4 (Theorem 3.1 of [FM12]). *Assume $\mathsf{OCA} + \mathsf{MA}_{\aleph_1}$. Suppose that*

$$\varphi : \mathcal{P}(\omega)/\text{Fin} \to \mathcal{C}(X)/\mathcal{K}(X)$$

is an injective homomorphism. Then φ is trivial.

Proof. Strategy of the proof is to show that $\mathcal{J}$ is a dense P-ideal and conclude by the Lemma 4.1.2. Suppose for a contradiction that $\mathcal{J}$ is not a dense P-ideal.

We will consider following cases independently: $\mathcal{J}$ is not a P-ideal; $\mathcal{J}$ is not dense.

We shall begin by the first case "$\mathcal{J}$ is not a P-ideal". By the Lemma 4.1.7 there is an uncountable almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A} \cap \mathcal{J} = \emptyset$. Therefore by the Lemma 4.2.2 there exists an uncountable subset $\mathcal{B} \subseteq \mathcal{A}$ and a coloring $b = b_0 \cup b_1$ for all $b \in \mathcal{B}$ such that $B_i = \{b_i : b \in \mathcal{B}\}$ are both tree-like for $i \in 2$. So by the Lemma 4.2.5, all but countably many of the elements of $\mathcal{B}_i$ belong to $\mathcal{J}$. Call it $\tilde{\mathcal{B}}_i$. Then by the pigeonhole principle there in uncountably many $b \in \mathcal{B}$ such that $b_i \in \tilde{\mathcal{B}}_i$ for both colors $i \in 2$. Since $\mathcal{J}$ is an ideal, $b_0, b_1 \in \mathcal{J}$ implies $b = b_0 \cup b_1 \in \mathcal{J}$. Thus $\mathcal{B} \cap \mathcal{J} \neq \emptyset$. Contradiction.

From the construction in Lemma 3.2.2, we know that there exists a sequence $\langle n_i : i \in \omega \rangle$ of integers and a sequence of subsets $\langle s_i \subseteq [n_i, n_{i+1}) : i \in \omega \rangle$ such that $\{a \subseteq \omega : a \cap [n_i, n_{i+1}) = s_i$ for infinitely many $i\}$ belongs to $\mathcal{J}$. Therefore $\mathcal{J}$ is comeagre so it is dense. □

Appendix A

Forcing Axioms and Combinatorics

The main results of this report depend on the set theoretic ambient. In other words, some phenomena in the scope of this work depend on the models of the set theory (*Zermelo-Fraenkel set theory with Axiom of Choice*, shortly ZFC). *Forcing Axioms* are higher versions of the Baire Category Theorem, ensuring the existence of as many generics as possible, and therefore providing models in which the universe is as complete as possible. We will state one here which will serve throughout this appendix: *Proper Forcing Axiom* (PFA).

Definition A.0.1. A forcing notion $\mathbb{P}$ is called *proper* if for every uncountable cardinal λ, forcing with $\mathbb{P}$ preserves stationary subsets of $[\lambda]^\omega$.

Definition A.0.2. PFA is the following statement: If $\mathbb{P}$ is a proper forcing notion and $\mathcal{D}$ is a family of $\aleph_1$ dense subsets of $\mathbb{P}$ then there is a $\mathcal{D}$-generic filter G on $\mathbb{P}$.

PFA was introduced by Baumgartner in [KV84, Chapter 21] although it was implicitly present in earlier work of Shelah [She82]. Its consistency was proved in [She82]. Properness is in fact a weakening of the ccc. Therefore in some sense PFA is a generalisation of Martin's Axiom. Yet unlike Martin's Axiom, consistency of PFA requires large cardinals. One may see [She16] or [She82] for a detailed study of PFA, its roots and consequences.

In this appendix we review some required notions and related results. For a general treatise of set theory see [Kun83], [Kun11] or [Jec03]. Our set theoretical notation follows these three books.

A.1 Martin's Axiom

We review some notions related to forcing and give the statement of *Martin's Axiom*. One may refer to [Fre08] for an extensive study of consequences of

Martin's Axiom.

Definition A.1.1. A *forcing poset* is a triple $(\mathbb{P}, \leq, 1)$ such that $\leq$ is a preorder with largest element 1.

Definition A.1.2. Let $\mathbb{P}$ be a forcing poset, $p, q \in \mathbb{P}$ and $A \subseteq \mathbb{P}$. Then

1. p, q are *comparable* if $p \leq q$ or $q \leq p$;
2. p, q are *compatible* (denoted by $p \parallel q$) if there is some $r \in \mathbb{P}$ such that $r \leq p, q$;
3. p, q are *incompatible* (denoted by $p \perp q$) if they are not compatible,
4. A is a *chain* if elements of A are pairwise comparable;
5. A is an *antichain* if elements of A are pairwise incompatible;
6. A is *dense* in $\mathbb{P}$ if for any $x \in \mathbb{P}$ there exists $y \in A$ such that $y \leq x$;
7. $\mathbb{P}$ has the *countable chain condition* (ccc) if every antichain is countable.

Definition A.1.3. $G \subseteq \mathbb{P}$ is a *filter* if

1. $1 \in G$,
2. for any $x, y \in G$, there exists $r \in G$ such that $r \leq x, y$ and
3. for any $x, y \in \mathbb{P}$, if $x \in G$ and $x \leq y$ then $y \in G$.

Definition A.1.4. MA_κ is the following statement: For every ccc poset $\mathbb{P}$, whenever $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

Proofs of the following theorems can be found in [Kun83, Chapters II and VIII.6] or alternatively in [Kun11, Chapters III and V.6]. Relative consistency results can be achieved by iterative forcing (as it is done in Kunen) as well as using relative the consistency of PFA.

Theorem A.1.5. MA_κ *is relatively consistent with* ZFC. □

Theorem A.1.6. MA_κ *implies that the continuum is strictly bigger than* κ. □

Therefore Martin's axiom is independent from ZFC and CH is inconsistent with it.

A.2 Open Coloring Axiom

Open Coloring Axiom (OCA) is a Ramsey type axiom introduced by Stevo Todorčević in [Tod89]. In the same monograph, it is proven that OCA follows from PFA. In particular this result implies that $\mathsf{MA}_{\aleph_1}$ + OCA is relatively consistent. Now fix a separable metric space X.

Notation A.2.1. Let S, S' be two sets. Then $[S]^{S'}$ stands for the set $\{s \subseteq S : |s| = |S'|\}$.

Definition A.2.2. A partition (or interchangeably a coloring) $[X]^2 = M_0 \cup M_1$ is *open* if the set $\tilde{M_0} = \{(a, b), (b, a) \in X \times X : \{a, b\} \in M_0\}$ is open in the product topology $X \times X \setminus$ diagonal.

Definition A.2.3. OCA is the following statement: Let $[X]^2 = M_0 \cup M_1$ be an open coloring. Then either X has an uncountable M_0-homogeneous subset or it can be covered by a countable family of M_1-homogeneous subsets.

Theorem A.2.4 (Theorem 8.0 in [Tod89])**.** PFA *implies* OCA. □

Note that OCA can also be proven in ZFC i.e. without assuming existence of large cardinals (see [Vel92]).

A.3 Combinatorics

In this section we will discuss some infinitary combinatorics. For more combinatorial results related to the subject see [Far19].

Let us begin our discussion with stating the most basic combinatorial method, the so-called *pigeonhole principle*. Let $\kappa < \lambda$ be two infinite cardinals. If $\lambda = \bigcup_{\alpha\in\kappa} S_\alpha$ then, by the Axiom of Choice, $|S_\alpha| = \lambda$ at least for one $\alpha \in \kappa$.

Definition A.3.1. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called *almost disjoint* if for any distinct $a, b \in \mathcal{F}$ we have $a \cap b \in \mathrm{Fin}$.

For $a, b \in \mathcal{P}(\omega)$, we say b *almost includes* a if $a \setminus b \in \mathrm{Fin}$. This is denoted by $a \subseteq^* b$. Similarly we say b is *almost equal* to a if $a \triangle b \in \mathrm{Fin}$. This is denoted by $a =^* b$. It is easy to observe that $(\mathcal{P}(\omega), \subseteq^*)$ is a *quasi order.*

An order $(X, <)$ is called σ-directed if for every countable $A \subseteq X$ there exists $a \in X$ such that $a < x$ for all $a \in A$.

Proposition A.3.2. *Let $\langle b_n : n \in \omega\rangle$ be a sequence of subsets of ω. Let $\mathcal{A}$ be the set of all subsets of ω which are almost disjoint from each b_n. Then $(\mathcal{A}, \subseteq^*)$ is σ-directed.*

Proof. Take without loss of generality an increasing sequence $\langle c_n : n \in \omega \rangle$ of $\mathcal{A}$. If $C = \bigcup_{n\in\omega} c_n \in \mathcal{A}$ then there is nothing to do. Otherwise, $C \cap b_{n_j}$ is infinite for at most countably many n_j. The finite case is trivial since $C \setminus \bigcup_i b_{n_j}$ is clearly in $\mathcal{A}$. Suppose that there are infinitely many n_j. Now by eliminating the elements from c_i which are also in b_{n_j} diagonally we get $C' = \bigcup_{i\in\omega} c_i \setminus \bigcup_{j\leq i} b_{n_j}$ which is clearly in $\mathcal{A}$ and $c_n \subseteq^* C'$ for all $n \in \omega$. □

Proposition A.3.3 (Δ-System lemma)**.** *Let $\mathcal{F} = \{A_\alpha : \alpha \in \omega_1\}$ be an uncountable family of finite sets. Then there is an uncountable $W \subseteq \omega_1$ and a finite set R such that $A_\alpha \cap A_\beta = R$ for any two distinct $\alpha, \beta \in W$.*

Proof. We start by an elementary observation:

Claim A.3.4. We may assume without loss of generality that all A_α have the same size.

Proof. Considering the partition $\mathcal{F} = \bigcup_{k\in\omega}\{A_\alpha : |A_\alpha| = k\}$, we see immediately that there is at least one $n \in \omega$ such that $\{A_\alpha : |A_\alpha| = n\}$ is uncountable by the pigeonhole principle. □

We will argue by induction on the size of elements of $\mathcal{F}$. For $n = 1$, we have clearly $W = \omega_1$ and $R = \emptyset$. Induction hypothesis: for $n = k$, there exist desired $W \subseteq \omega_1$ and R. Let us examine the case $n = k + 1$.

Suppose there exists an x such that $x \in A_\alpha$ for uncountably many α. Then consider the family of $A_\alpha \setminus \{x\}$, whose elements are of size k. By the induction hypothesis, there exist an uncountable $W' \subseteq \omega_1$ and a finite R' such that $(A_\alpha \setminus \{x\}) \cap (A_\beta \setminus \{x\}) = R'$ for all $\alpha \neq \beta \in W'$. Therefore for $W = W'$ and $R = R' \cup \{x\}$ we have $A_\alpha \cap A_\beta = R$ for all $\alpha \neq \beta \in W$.

Now suppose that for all x we have $x \in A_\alpha$ for countably many α. Then we can construct a disjoint uncountable subfamily $\{B_\alpha : \alpha \in \omega_1\} \subseteq \mathcal{F}$ by induction on α. Let $B_0 = A_0$. Suppose we have constructed B_β for all $\beta < \alpha$. As α is countable, we made at most $\aleph_0^n = \aleph_0$ choice of elements for these B_β. So there is some $\gamma \in \omega_1$ such that $A_\gamma \cap B_\beta = \emptyset$ for all $\beta < \alpha$. Choose $B_\alpha = A_\gamma$. □

Definition A.3.5. Let $\mathbb{P}$ be a poset. The *bounding number* of $\mathbb{P}$ is the minimal cardinal of an unbounded subset of $\mathbb{P}$. It is denoted by $\mathfrak{b}_\mathbb{P}$. In particular $\mathfrak{b}$ denotes the bounding number of $(\omega^\omega, \leq^*)$.

Proposition A.3.6. MA_κ *implies that* $\mathfrak{b} > \kappa$.

Proof. Suppose for a contradiction that there exists some unbounded family $\mathcal{F}$ of cardinal κ. Define the forcing notion $\mathbb{P}$ as follows: $p \in \mathbb{P}$ if and only if $p = (F_p, A_p, f_p)$ where

- $F_p \in [\mathcal{F}]^{<\omega}$,

- $A_p \in \text{Fin}$ and
- $f_p : A_p \to \omega$ dominating all $g \in F_p$ on A_p

equipped with the order $p \leq q$ if and only if

- $F_q \subseteq F_p$,
- $A_q \subseteq A_p$ and
- $f_q \leq f_p$.

Claim A.3.7. $\mathbb{P}$ is ccc.

Proof. Indeed $\mathbb{P}$ is more than ccc : take any two distinct $p, q \in \mathbb{P}$. Then we can easily find $r \leq p, q$ where $F_r = F_p \cup F_q$, $A_r = A_p \cup A_q$ and $f_r : A_p \cup A_q \to \omega$ dominating F_r on $A_p \cup A_q$. □

Now define $D_n = \{p : n \in A_p\}$ where $n \in \omega$ and $E_f = \{p : f \in F_p\}$ where $f \in \mathcal{F}$. Observe that D_n and E_f are dense in $\mathbb{P}$. Then MA_κ implies that there is a $\{D_n, E_f : n \in \omega \text{ and } f \in \mathcal{F}\}$-generic filter G. Therefore $f = \bigcup\{f_p : p \in G\}$ is a total function dominating $\mathcal{F}$. This contradicts the hypothesis. □

Appendix B

Stone Spaces and Corona Spaces

One of the main objects of this report is the Stone-Čech compactification. It is defined by a universal property for a certain category of topological spaces. In this appendix we will first review the construction of the Stone-Čech compactification, Stone Duality and their relation for the zero dimensional locally compact spaces. Then we will study Parovičenko's characterization of ω^*.

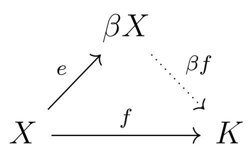
B.1 Stone-Čech Compactification

The Stone-Čech compactification β is a functor which associates a topological space X to a compact Hausdorff space βX such that any continuous map from X to a compact Hausdorff space factors through βX. Intuitively this is the "largest" possible compactification satisfying this universal property. In this section we will construct the Stone-Čech compactification of topological spaces. We will omit most of the proofs which can be found in the first chapter of [Wal74] or in the second chapter of [CN74].

Definition B.1.1. Let X be a topological space. The pair (K, e) is a *compactification* of X if K is a compact space and $e : X \to K$ is a dense embedding.

Definition B.1.2. Let X be a topological space. A *Stone-Čech compactification* of X is $(\beta X, e)$ such that:

- $(\beta X, e)$ is a compactification of X where βX is Hausdorff,
- the universal property below is satisfied : for any compact Hausdorff space K and for any continuous function $f : X \to K$, there exists a unique continuous function $\beta f : \beta X \to K$ such that the diagram



commutes.

Remark B.1.3. The Stone-Čech compactification of a completely regular space is unique up to homeomorphism.

Notation B.1.4. Let X be a topological space. We let $C(X)$ denote the ring of continuous functions $f : X \to \mathbb{C}$ and $C^*(X)$ denote the subring of bounded functions.

Definition B.1.5. A *zero-set* of X is a set equal to $f^{-1}(\{0\})$ for some $f \in C(X)$. We will denote the family of zero-sets of X by $\mathfrak{Z}(X)$.

The filters (respectively ultrafilters) on $\mathfrak{Z}(X)$ are called the *z-filters* (respectively *z-ultrafilters*) in the literature. There are several equivalent constructions of the Stone-Čech compactification. We will give the one which uses z-ultrafilters.

Note that the definition of the Stone-Čech compactification could equivalently be made in terms of complex valued continuous functions as follows: $(\beta X, e)$ is a compactification of X where e is a C^* embedding (every bounded function on X extends continuously to βX). Tychonoff's characterisation of *completely regular* topological spaces

> X is completely regular if and only if there is an embedding of X in a product of copies of the closed unit interval

strongly suggests that this class is the appropriate class to study through the compactifications when we construct the Stone-Čech compactification as $[0, 1]^{C(X)}$ (see pg.8 in [Wal74]). We can reformulate Tychonoff's characterisation to build a bridge between the construction mentioned above and the z-ultrafilter one, as follows:

> X is completely regular if and only if $\mathfrak{Z}(X)$ is a basis for the closed sets of X.

Theorem B.1.6 (Čech, Stone)**.** *Every completely regular space has a Stone-Čech compactification.* □

The construction by z-ultrafilters is as follows: βX is the set of all z-ultrafilters on X with the *Stone topology* determined by the basis

$$\mathcal{B} = \{\{u \in \beta X : A \notin u\} : A \in \mathfrak{Z}(X)\}.$$

And the *canonical embedding* $e : X \to \beta X$ is given by $x \to u_x$ where u_x is the principal ultrafilter generated by x. One may check by hand (see pg. 24 in [CN74]) that this construction works.

By abuse of notation $e(X) \subseteq \beta X$ is traditionally identified with X. This is the "trivial" part of the compactification. The rest $\beta X \setminus X$ is called the *Stone-Čech remainder* and denoted X^*. A priori we do not know whether X^* is compact or not. The following result, whose proof can be found in [CN74] (Lemma 2.9), gives a definitive answer to this.

Theorem B.1.7. *X^* is compact if and only if X is locally compact.* □

B.2 Stone Duality

We assume that the reader is familiar with the definition and basic facts about Boolean algebras. These can be found in the extensive work [Hal74]. In this section we will set up the Stone Duality between Boolean algebras and *Boolean spaces* (zero-dimensional compact Hausdorff spaces), and relate this to the Stone-Čech compactification of totally disconnected zero dimensional spaces. We will state the theorems without proofs which can be found in the second chapter of [Wal74] or [CN74].

Let X be a topological space. Then clopen subsets of X form a Boolean algebra denoted $\mathcal{C}(X)$.Therefore the more connected the space X is the smaller $\mathcal{C}(X)$.

Generalised Cantor space 2^I (with the product topology) are the prototypes of Boolean spaces. Also note that any product of Boolean spaces is Boolean.

Definition B.2.1. Let $\mathcal{B}$ be a Boolean algebra. The *Stone Space* or the *dual space* of $\mathcal{B}$ is the set of all ultrafilters $S(\mathcal{B})$ of $\mathcal{B}$ with the topology generated by the basis

$$\{s(a) : a \in \mathcal{B}\}$$

where $s(a) = \{u \in S(\mathcal{B}) : a \in u\}$.

The map

$$\begin{aligned} s : \mathcal{B} &\to \mathcal{P}(S(\mathcal{B})) \\ a &\mapsto s(a) \end{aligned}$$

is called the *Stone map*

Theorem B.2.2 (Stone Representation Theorem (algebraic))**.** *Let $\mathcal{B}$ be a Boolean algebra. Then $S(\mathcal{B})$ is Boolean and $s : \mathcal{B} \to \mathcal{C}(S(\mathcal{B}))$ is an isomorphism.* □

Let X be a Boolean space. Then $\mathcal{C}(X)$ is the *dual algebra* of X. Define $t(x) = \{a \in \mathcal{C}(X) : x \in a\}$. Clearly

$$\begin{aligned} t : X &\to S(\mathcal{C}(X)) \\ x &\mapsto t(x) \end{aligned}$$

is well defined.

Theorem B.2.3 (Stone Representation Theorem (topological)). *Let X be a Boolean space. Then t is a homeomorphism.* □

Therefore the bidual of a Boolean algebra/space is itself.

Theorem B.2.4. *Let X be a Boolean space and $\mathcal{A}$ be a Boolean algebra. Then $\{x \in 2^X : \text{ continuous}\}$ is a subalgebra of 2^X isomorphic to $\mathcal{C}(X)$ and $\{x \in 2^{\mathcal{A}} : \text{ homomorphism}\}$ is a closed subspace of $2^{\mathcal{A}}$ homeomorphic to $S(\mathcal{A})$.* □

Now we will study the homomorphisms and continuous mappings and see that Stone Duality is indeed a contravariant functor between the categories of Boolean algebras and of Boolean Spaces.

Let $f : \mathcal{A} \to \mathcal{B}$ be a Boolean algebra homomorphism. We define the *dual* of f by

$$f^d : S(\mathcal{B}) \to S(\mathcal{A}).$$
$$u \mapsto f^{-1}(u)$$

f^d is well defined since preimages of ultrafilters under morphisms are ultrafilters. And similarly we define the dual of a continuous map $\varphi : X \to Y$ of Boolean spaces by

$$\varphi^d : \mathcal{C}(Y) \to \mathcal{C}(X).$$
$$a \mapsto f^{-1}(a)$$

Theorem B.2.5 (Stone Representation Theorem (morphisms)). *Let $f : \mathcal{A} \to \mathcal{B}$ be a Boolean algebra homomorphism and $\varphi : X \to Y$ be a continuous map of Boolean spaces. Then*

1. *f^d is continuous,*
2. *φ^d is a homomorphism*
3. *f is one-to one if and only if f^d is onto,*
4. *φ is one-to one if and only if φ^d is onto and*
5. *diagrams below commute.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{s_A} & \mathcal{C}(S(\mathcal{A})) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^{dd}} \\ \mathcal{B} & \xrightarrow{s_B} & \mathcal{C}(S(\mathcal{B})) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{t_X} & S(\mathcal{C}(X)) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi dd} \\ Y & \xrightarrow{t_Y} & S(\mathcal{C}(Y)) \end{array}$$

□

This observation with the Stone Representation Theorem leads to the following important results for zero-dimensional totally disconnected spaces:

Theorem B.2.6. *Let X be a totally disconnected zero-dimensional space. Then $\beta t : \beta X \to S(\mathcal{C}(X))$ is a homeomorphism.* □

Theorem B.2.7. *X is strongly zero-dimensional if and only if βX is zero-dimensional.* □

Theorem B.2.8. *Let X be a locally compact, zero-dimensional space. Set $\mathcal{F}(X) = \{A \in \mathcal{C}(X) : A \text{ is cocompact}\}$ and $\hat{A} = cl_{\beta X} A \cap X^*$ for all $A \in \mathcal{C}(X)$. Then $A \mapsto \hat{A}$ is a Boolean algebra homomorphism from $\mathcal{C}(X)$ onto $\mathcal{C}(X^*)$ with associated filter $\mathcal{F}(X)$, and $\mathcal{C}(X^*)$ is isomorphic to $\mathcal{C}(X)/\mathcal{F}(X)$.* □

As a conclusion of this section observe that the dual algebra of $\beta\omega$ is $\mathcal{P}(\omega)$, ω^* is ω/Fin, βX is $\mathcal{C}(X)$ and X^* is $\mathcal{C}(X)/\mathcal{K}(X)$ for zero-dimensional, locally compact X. Moreover any continuous map $\varphi : X^* \to Y^*$ corresponds to $\varphi^d : \mathcal{C}(Y)/\mathcal{K}(Y) \to \mathcal{C}(X)/\mathcal{K}(X)$.

B.3 Parovičenko's characterization of ω^*

> *"The space $\beta\omega$ is a monster with three heads. If one works in a model in which the Continuum Hypothesis holds, then one will see only the first head. This head is smiling, friendly, and makes you feel comfortable working with $\beta\omega$."*
>
> Jan van Mill

In this section we will see how X^* behaves under CH. Parovičenko showed in his 1963 paper that, for any X from a relevant class[1] CH implies that X^* is homeomorphic to ω^*. This section is based on [KV84, Chapter 11] and [Wal74, Chapter 3].

Let $\mathcal{B}$ be a Boolean algebra. We say that $\mathcal{B}$ satisfies *condition H_ω* if for all $F \in [\mathcal{B} \setminus \{1\}]^{\leq\omega}$ and $G \in [\mathcal{B} \setminus \{0\}]^{\leq\omega}$ with $F < G$ (i.e for any finite $F' \subseteq F$ and $G' \subseteq G$ we have $\vee F', \wedge G'$) there is an element $x \in \mathcal{B}$ such that $F < \{x\} < G$.

From now on X refers to a zero-dimensional, locally compact, noncompact Polish space. For the proof of the following lemma we will adopt an operator algebras point of view rather than a topological one. Recall that such X is σ compact as $X = \bigcup_n K_n$. Define $X_0 = K_0$ and $K_n = K_n \setminus K_{n-1}$. We will identify $\mathcal{C}(X)$ with $\prod_n \mathcal{C}(X_n)$ and $\mathcal{K}(X)$ with $\bigoplus_n \mathcal{C}(X_n)$.

Lemma B.3.1. *$\mathcal{C}(X)/\mathcal{K}(X)$ satisfies H_ω.*

[1] So-called *Parovičenko spaces.*

Sketch of the proof. Let $F, G \subseteq \mathcal{C}(X)/\mathcal{K}(X) \setminus \{0,1\}$ be two countable families satisfying $F < G$. Enumerate F and G as $\{f_n : n \in \omega\}$ and $\{g_n : n \in \omega\}$ respectively. Assume without loss of generality that $f_0 < f_1 < \cdots$ and $g_0 > g_1 > \cdots$. Take representatives $F_n, G_n \in \mathcal{C}(X)$ of f_n, g_n. For all $j \in \omega$ there is a $k \in \omega$ such that for all $l \geq k$ we can find a h_l satisfying $\pi_l(f_i) \subseteq h_l \subseteq \pi_l(g_i)$ for all $n \leq j$ where π_l are the projections. We find the desired h with $F < \{h\} < G$ by diagonalising h_l. □

The following corollary is therefore a trivial special case.

Corollary B.3.2. $\mathcal{P}(\omega)/\mathrm{Fin}$ *satisfies* H_ω.

Definition B.3.3. Let $\mathcal{B}$ be a Boolean algebra. We say that $\mathcal{B}$ satisfies *condition* R_ω if for any $F \in [\mathcal{B} \setminus \{1\}]^{\leq\omega}$, $G \in [\mathcal{B} \setminus \{0\}]^{\leq\omega}$ and $H \in [\mathcal{B}]^{\leq\omega}$ such that

1. $F < G$,
2. $\forall F' \in [F]^{<\omega}\, \forall G' \in [G]^{<\omega}\, \forall h \in H : h \not\leq F'$ and $\wedge G' \not\leq h$,
3. $F < \{x\} < G$,
4. $\forall h \in H : h \not\leq x$ and $x \not\leq h$.

The following lemma will be required for the proof of the main theorem.

Lemma B.3.4. *If a Boolean algebra $\mathcal{B}$ satisfies condition H_ω, then it satisfies condition R_ω.*

Proof. Let F, G and H be as in Definition B.3.3 (1) and (2). Enumerate $F = \{f_n : n \in \omega\}$, $G = \{g_n : n \in \omega\}$ and $H = \{h_n : n \in \omega\}$. For each $h \in H$ and finite $F' \subseteq F$ we have that $(\vee F')^c \wedge h \neq 0$, therefore there exists, by applying the condition H_ω for all $n \in \omega$, an element $d_n \in \mathcal{B} \setminus \{0\}$ such that $d_n < h_n$ and $f \wedge d_n = 0$ for all $f \in F$. Similarly we can find $e_n \in \mathcal{B} \setminus \{0\}$ such that $\{e_n\} < G$ and $e_n \wedge h_n = 0$.

We can indeed assure that $e_n \wedge d_m = 0$ for all $n, m \in \omega$. Now define for all $n \in \omega$, $\tilde{f}_n = f_n \vee e_n$ and $\tilde{g_n} = g_n \wedge d_n^c$.

Observe that for any $n, m \in \omega$ we have $\bigvee_{0 \leq i \leq n} \tilde{f}_i \leq \bigwedge_{0 \leq j \leq n} \tilde{g}_j$. By H_ω we can find $x \in \mathcal{B}$ such that for all $n, m \in \omega$

$$\bigvee_{0 \leq i \leq n} \tilde{f}_i \leq x \leq \bigwedge_{0 \leq j \leq n} \tilde{g}_j.$$

□

Now we can state and prove the main result of this section. The sketch of the proof is due to van Mill.

Theorem B.3.5 (Parovičenko's Theorem)**.** *Assume* CH*. If* $\mathcal{B}$ *is a Boolean algebra of cardinality at most* $\mathfrak{c}$ *satisfying* H_ω *then* $\mathcal{B}$ *is isomorphic to* $\mathcal{P}(\omega)/\mathrm{Fin}$*.*

Proof. Let $\mathcal{B}$ and $\mathcal{C}$ be two Boolean algebras satisfying H_ω such that $|\mathcal{B}|, |\mathcal{C}| \leq \mathfrak{c}$. By CH enumerate $\mathcal{B} = \{b_\alpha : \alpha \in \omega_1\}$ and $\mathcal{C} = \{c_\alpha : \alpha \in \omega_1\}$. Without loss of generality assume that $b_0 = 0$ and $c_0 = 0$. We will reason by "back and forth". By transfinite induction on α we will construct subalgebras $\mathcal{B}_\alpha \subseteq \mathcal{B}$, $\mathcal{C}_\alpha \subseteq \mathcal{C}$ and isomorphisms $\sigma_\alpha : \mathcal{B}_\alpha \to \mathcal{C}_\alpha$ such that

1. $b_\alpha \in \mathcal{B}_\alpha$ and $c_\alpha \in \mathcal{C}_\alpha$,
2. if $\beta < \alpha$ then $\mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$, $\mathcal{C}_\beta \subseteq \mathcal{C}_\alpha$ and $\sigma \restriction \mathcal{B}_\beta = \sigma_\beta$.

Let $\mathcal{B}_0 = \mathcal{C}_0 = \{0, 1\}$ and σ_0 be defined canonically. Suppose that $\mathcal{B}_\beta, \mathcal{C}_\beta$ and σ_β are defined for all $\beta < \alpha < \omega_1$ satisfying (1) and (2). If $b_\alpha \in \bigcup_{\beta\in\alpha} \mathcal{B}_\beta$ and $c_\alpha \in \bigcup_{\beta\in\alpha} \mathcal{C}_\beta$ then define $\mathcal{B}_\alpha = \bigcup_{\beta\in\alpha} \mathcal{B}_\beta$, $\mathcal{C}_\alpha = \bigcup_{\beta\in\alpha} \mathcal{C}_\beta$ and $\sigma_\alpha \in \bigcup_{\beta\in\alpha} \sigma_\beta$.

Now suppose without loss of generality $b_\alpha \notin \bigcup_{\beta\in\alpha} \mathcal{B}_\beta = \mathcal{F}$. Let $\sigma = \bigcup_{\beta\in\alpha} \sigma_\beta$. Put $\mathcal{F}_0 = \{f \in \mathcal{F} : f < b_\alpha\}$, $\mathcal{F}_1 = \{f \in \mathcal{F} : f > b_\alpha\}$ and $\mathcal{F}_2 = \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_1)$.

By Lemma B.3.4 there is an element $c \in \mathcal{C}$ such that $\sigma(\mathcal{F}_0) < \{c\} < \sigma(\mathcal{F}_1)$ and for all $c' \in \sigma(\mathcal{F}_2)$ we have $c' \not\leq c$ and $c \not\leq c'$. If we put $\sigma(b_\alpha) = c$ and $\sigma(b_\alpha^c) = c^c$ then σ can be extended to an isomorphism between the generated algebras $\tilde{\sigma} : \langle\langle \mathcal{F} \cup \{b_\alpha\} \rangle\rangle \to \langle\langle \sigma(\mathcal{F}) \cup \{c\} \rangle\rangle$. If $c_\alpha \notin \langle\langle \sigma(\mathcal{F}) \cup \{c\} \rangle\rangle$ we do the same thing in the converse direction. □

Bibliography

[CN74] W. W. Comfort and S. Negrepontis. *The theory of ultrafilters*, volume 211 of *Grundlehren Math. Wiss.* Springer, Cham, 1974.

[Far00] Ilijas Farah. *Analytic quotients. Theory of liftings for quotients over analytic ideals on the integers*, volume 702 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2000.

[Far19] Ilijas Farah. *Combinatorial set theory of C^*-algebras.* Springer Monogr. Math. Cham: Springer, 2019.

[FGVV22] Ilijas Farah, Saeed Ghasemi, Andrea Vaccaro, and Alessandro Vignati. Corona rigidity, 2022.

[FM12] Ilijas Farah and Paul McKenney. Homeomorphisms of cech-stone remainders: the zero-dimensional case. *Proceedings of the American Mathematical Society*, 146, 11 2012.

[Fre08] D. H. Fremlin. *Consequences of Martin's axiom*, volume 84 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, paperback reprint of the 1984 edition edition, 2008.

[Hal74] Paul R. Halmos. Lectures on Boolean algebras. Reprint. New York - Heidelberg - Berlin: Springer-Verlag. 147 p. (1974)., 1974.

[Jec03] Thomas Jech. *Set theory.* Springer Monogr. Math. Berlin: Springer, the third millennium edition, revised and expanded edition, 2003.

[JN76] S.A Jalali-Naini. *The Monotone Subsets of Cantor Space, Filters and Descriptive Set Theory.* Oxford, 1976.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Grad. Texts Math.* Berlin: Springer-Verlag, 1995.

[Kun83] Kenneth Kunen. *Set theory. An introduction to independence proofs. 2nd print*, volume 102 of *Stud. Logic Found. Math.* Elsevier, Amsterdam, 1983.

[Kun11] Kenneth Kunen. *Set theory.*, volume 34 of *Stud. Log. (Lond.).* London: College Publications, 2011.

[KV84] Kenneth Kunen and Jerry E. Vaughan. Handbook of set-theoretic topology. Amsterdam-New York-Oxford: North-Holland. VII, 1273 p. $ 98.00; Dfl. 275.00 (1984)., 1984.

[MS70] D. A. Martin and R. M. Solovay. Internal Cohen extensions. *Ann. Math. Logic*, 2:143–178, 1970.

[MV21] Paul McKenney and Alessandro Vignati. Forcing axioms and coronas of C*-algebras. *J. Math. Log.*, 21(2):73, 2021. Id/No 2150006.

[Par63] I. I. Parovichenko. A universal bicompact of weight S. *Sov. Math., Dokl.*, 4:592–595, 1963.

[Rud56] Walter Rudin. Homogeneity problems in the theory of Cech compactifications. *Duke Math. J.*, 23:409–419, 1956.

[She82] Saharon Shelah. *Proper forcing*, volume 940 of *Lect. Notes Math.* Springer, Cham, 1982.

[She16] Saharon Shelah. *Proper and improper forcing.* Perspect. Log. Cambridge: Cambridge University Press; Urbana, IL: Association for Symbolic Logic (ASL), 2nd edition. Reprint of the 1998 original published by Springer edition, 2016.

[SS88] Saharon Shelah and Juris Steprāns. PFA implies all automorphisms are trivial. *Proc. Am. Math. Soc.*, 104(4):1220–1225, 1988.

[Tal80] Michel Talagrand. Compacts de fonctions mesurables et filtres non mesurables. *Stud. Math.*, 67:13–43, 1980.

[Tod89] Stevo Todorčević. *Partition problems in topology*, volume 84 of *Contemp. Math.* Providence, RI: American Mathematical Society, 1989.

[Vel92] Boban Velickovic. Applications of the open coloring axiom. In *Set theory of the continuum. Papers presented at the Mathematical Sciences Research Institute (MSRI) workshop, Berkeley, California, USA, October 16-20, 1989*, pages 137–154. New York: Springer-Verlag, 1992.

[Vel93] Boban Velickovic. OCA and automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$. *Topology Appl.*, 49(1):1–13, 1993.

[Vig22] Alessandro Vignati. Rigidity conjectures for continuous quotients. *Ann. Sci. Éc. Norm. Supér. (4)*, 55(6):1687–1738, 2022.

[Wal74] Russell C. Walker. *The Stone-Cech compactification*, volume 83 of *Ergeb. Math. Grenzgeb.* Springer-Verlag, Berlin, 1974.